Extension 2 2009 Solution

Q1

(a) $\int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2 + C$.

(b) $\int xe^{2x}dx$.

Let $u = x, du = dx$; let $dv = e^{2x}dx, v = \frac{1}{2}e^{2x}$.

$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

(c) $\int \frac{x^2}{1+4x^2}dx = \frac{1}{4}\int \frac{4x^2+1-1}{1+4x^2}dx$

$= \frac{1}{4}\left(x - \frac{1}{2}\tan^{-1}(2x)\right) + C$.

(d) $\int_2^5 \frac{x-6}{x^2+3x-4}dx = \int_2^5 \frac{x-6}{(x+4)(x-1)}dx$

$= \int_2^5 \left(\frac{2}{x+4} + \frac{-1}{x-1}\right)dx = \left[2\ln(x+4) - \ln(x-1)\right]_2^5$

$= 2\ln\frac{9}{6} - \ln\frac{4}{1} = \ln\frac{9}{4} - \ln 4 = \ln\frac{9}{16}$.

(e) $\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}dx = \int_1^{\sqrt{3}} \frac{1}{x^3\sqrt{\frac{1+x^2}{x^2}}}dx$

$= \int_1^{\sqrt{3}} \frac{1}{x^3\sqrt{\frac{1+x^2}{x^2}}}dx = \int_1^{\sqrt{3}} \frac{1}{x^3\sqrt{\frac{1}{x^2}+1}}dx.$

Let $u = \frac{1}{x^2}, du = -\frac{2}{x^3}dx.$

When $x = 1, u = 1$; when $x = \sqrt{3}, u = \frac{1}{3}$.

$I = -\frac{1}{2}\int_1^{\frac{1}{3}} \frac{du}{\sqrt{u+1}} = \left[\sqrt{u+1}\right]_{\frac{1}{3}}^1$

$= \sqrt{2} - \sqrt{\frac{4}{3}} = \sqrt{2} - \frac{2}{\sqrt{3}} = \frac{\sqrt{6}-2}{\sqrt{3}}.$

Alternatively,

Let $x = \tan\theta, dx = \sec^2\theta d\theta$.

When $x = 1, \theta = \frac{\pi}{4}$; when $x = \sqrt{3}, \theta = \frac{\pi}{3}$.

$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\tan^2\theta\sec\theta}d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}d\theta =$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}d\theta = \left[-\frac{1}{\sin\theta}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sqrt{2} - \frac{2}{\sqrt{3}}.$

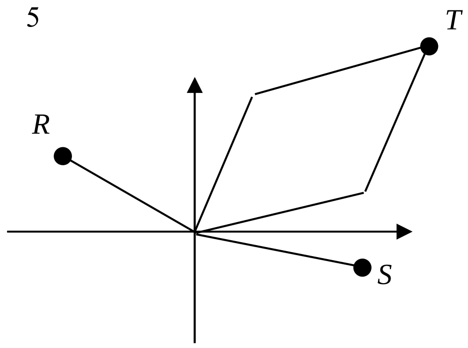
Q2

(a) $i^9 = i^8.i = i$, since $i^4 = 1$.

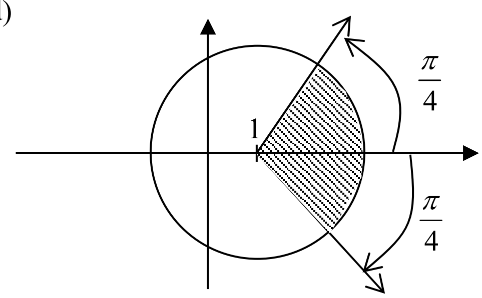
(b) $\frac{-2+3i}{2+i} = \frac{-2+3i}{2+i} \times \frac{2-i}{2-i} = \frac{-4+3+6i+2i}{5}$

$= \frac{-1+8i}{5}$

(c)



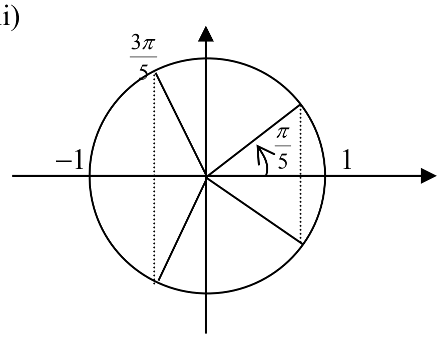
(d)



(e) (i) $\sqrt[5]{-1} = \sqrt[5]{\text{cis}(\pi + 2k\pi)} = \text{cis}\frac{\pi+2k\pi}{5}, k = 0, \pm1, \pm2.$

$\therefore \sqrt[5]{-1} = \text{cis}\frac{\pm\pi}{5}, \text{cis}\frac{\pm3\pi}{5}, \text{cis}\,\pi \ (= -1).$

(ii)



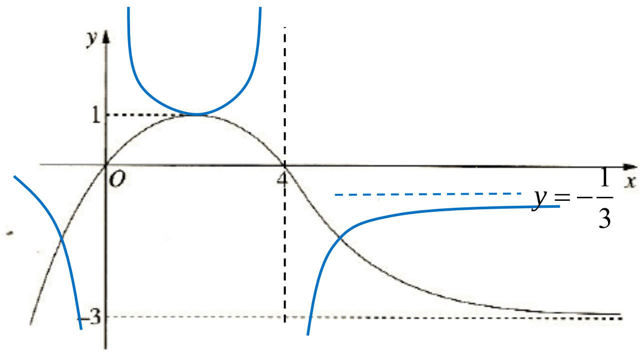
(f) (i) $\sqrt{3+4i} = \sqrt{2^2 - 1^2 + 2\times2\times1\times i} = \pm(2+i)$

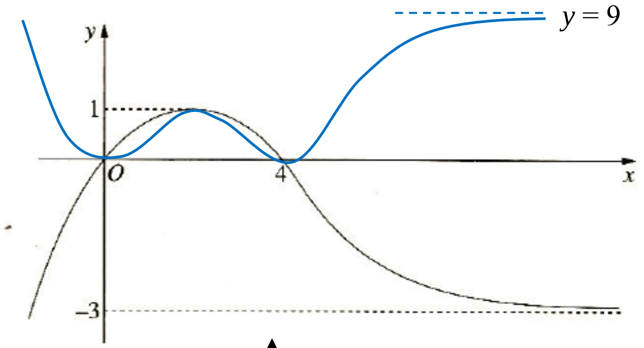
(ii) $z = \frac{-i \pm \sqrt{i^2 + 4(1+i)}}{2} = \frac{-i \pm \sqrt{3+4i}}{2}$

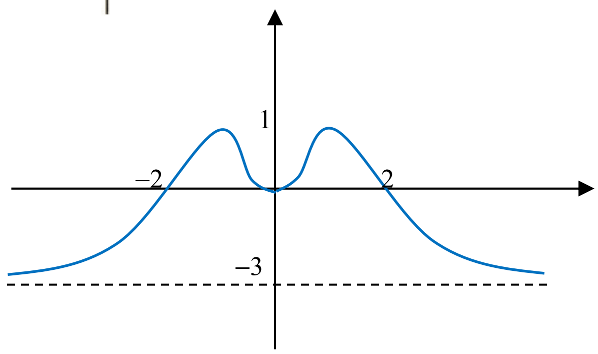
$= \frac{-i \pm (2+i)}{2} = \frac{2}{2} = 1$ or $\frac{-2-2i}{2} = -1-i$.

Q3

(a)







(b) $2x+2\left(y+xy'\right)+6yy'=0$.

$x+y+y'(x+3y)=0.$

$y'=-\dfrac{x+y}{x+3y}.$

Horizontal tangent, $\therefore y'=0, \therefore y=-x.$

Sub. to the original equation,

$x^2-2x^2+3x^2=18.$

$2x^2=18.$

$x^2=9.$

$x=\pm 3.$

$\therefore$ The points are $(3,-3),(-3,3)$.

(c) $P'(x)=3x^2+2ax+b.$

$P(1)=0, \therefore 1+a+b+5=0, \therefore a+b+6=0.$

$P'(1)=0, \therefore 3+2a+b=0.$

Solving simultaneous equations gives $a=3, b=-9.$

(d) $x+1=(x-1)^2.$

$x+1=x^2-2x+1.$

$-x^2+3x=0.$

$\therefore x=0,3.$

$V=2\pi\int_0^3 x(y_2-y_1)dx=2\pi\int_0^3 x\left(x+1-(x-1)^2\right)dx$

$=2\pi\int_0^3 x\left(-x^2+3x\right)dx=2\pi\int_0^3\left(-x^3+3x^2\right)dx$

$=2\pi\left[-\dfrac{x^4}{4}+x^3\right]_0^3=\dfrac{27\pi}{2}$ units^3.

Q4

(a)

(i) $\dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0.$

$y'=-\dfrac{b^2x}{a^2y}.$

$m_1=-\dfrac{b^2x_0}{a^2y_0}, \therefore m_2=\dfrac{a^2y_0}{b^2x_0}.$

The equation of the normal is

$y-y_0=\dfrac{a^2y_0}{b^2x_0}(x-x_0).$

(ii) Let $y=0$,

$-y_0=\dfrac{a^2y_0}{b^2x_0}(x-x_0).$

$x-x_0=-\dfrac{b^2x_0}{a^2}.$

$x=x_0-\dfrac{b^2x_0}{a^2}=x_0\left(1-\dfrac{b^2}{a^2}\right)=x_0\left(1-\dfrac{a^2(1-e^2)}{a^2}\right)$

$=e^2x_0.$

$\therefore N\left(e^2x_0,0\right).$

(iii) $\dfrac{PS}{PS'}=\dfrac{ePM}{ePM'}=\dfrac{\frac{a}{e}-x_0}{x_0+\frac{a}{e}}=\dfrac{a-ex_0}{a+ex_0}.$

$\dfrac{NS}{NS'}=\dfrac{ae-e^2x_0}{e^2x_0+ae}=\dfrac{a-ex_0}{ex_0+a}.$

$\therefore \dfrac{PS}{PS'}=\dfrac{NS}{NS'}.$

(iv) In $\Delta PNS'$, $\dfrac{\sin\alpha}{\sin\angle PNS'}=\dfrac{NS'}{PS'}.$

In ΔPNS, $\dfrac{\sin\beta}{\sin\angle PNS}=\dfrac{NS}{PS}.$

But $\angle PNS=\pi-\angle PNS', \therefore \sin\angle PNS=\sin\angle PNS'$,

and since $\dfrac{PS}{PS'}=\dfrac{NS}{NS'}, \therefore \dfrac{NS}{PS}=\dfrac{NS'}{PS'}.$

$\therefore \sin\alpha=\sin\beta.$

$\therefore \alpha=\beta$, since $\alpha+\beta<\pi$.

(b)

(i) Resolving the forces on P,

vertically, $T\cos\alpha + N\sin\alpha = mg$, (1)

horizontally, $T\sin\alpha - N\cos\alpha = mr\omega^2$. (2)

(ii) $(1)\times\cos\alpha + (2)\times\sin\alpha$ gives

$T\left(\cos^2\alpha + \sin^2\alpha\right) = m\left(g\cos\alpha + r\omega^2\sin\alpha\right)$.

$\therefore T = m\left(g\cos\alpha + r\omega^2\sin\alpha\right)$.

$(1)\times\sin\alpha - (2)\times\cos\alpha$ gives

$N\left(\sin^2\alpha + \cos^2\alpha\right) = m\left(g\sin\alpha - r\omega^2\cos\alpha\right)$.

$\therefore N = m\left(g\sin\alpha - r\omega^2\cos\alpha\right)$.

(iii) When $T = N$,

$g\cos\alpha + r\omega^2\sin\alpha = g\sin\alpha - r\omega^2\cos\alpha$

$r\omega^2(\sin\alpha + \cos\alpha) = g(\sin\alpha - \cos\alpha)$.

$$\omega^2 = \frac{g(\sin\alpha - \cos\alpha)}{r(\sin\alpha + \cos\alpha)}$$

$= \frac{g}{r}\left(\frac{\tan\alpha - 1}{\tan\alpha + 1}\right)$, by dividing both top and bottom by $\cos\alpha$.

(iv) $\omega^2 > 0, \therefore \tan\alpha - 1 > 0, \therefore \tan\alpha > 1, \therefore \frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

Q5

(a)

(i) $\angle ADB = 90°$ (semi-circle angle)

$\angle ABD = 90° - \angle BAD$ (angle sum in ΔADB)

$\angle AKY = 90° - \angle BAD$ (angle sum in ΔAKY).

$\therefore \angle AKY = \angle ABD$.

Alternatively, prove $\Delta DXK \,|||\, \Delta XYB$.

(ii) $\angle AKX = \angle ABD$ (from above).

$\angle ABD = \angle ACD$ (angles subtending the same arc are equal).

$\therefore \angle AKX = \angle ACD$.

$\therefore CKDX$ is cyclic (angles subtending the same chord are equal).

(iii) $\angle ACK = 180° - \angle XDK$ (opposite angles in a cyclic quad are supplementary).

But $\angle XDK = 90°$, $\therefore \angle ACK = 90°$.

And $\angle ACB = 90°$ (semi-circle angle).

$\therefore B$, C and K are collinear.

Alternatively, since $\angle D = 90°$, KX is the diameter, $\therefore \angle KCX = 90°$.

(b)

(i) Let $u = x^{2n}, du = 2nx^{2n-1}$, and $dv = xe^{x^2}, v = \frac{1}{2}e^{x^2}$

$$I_n = \left[\frac{1}{2}x^{2n}e^{x^2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx$$

$$= \frac{e}{2} - nI_{n-1}.$$

(ii) $I_2 = \frac{e}{2} - 2I_1$.

$I_1 = \frac{e}{2} - I_0$

$$I_0 = \int_0^1 xe^{x^2}\,dx = \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1).$$

$$\therefore I_2 = \frac{e}{2} - 2\left[\frac{e}{2} - \frac{1}{2}(e-1)\right] = \frac{e}{2} - 1.$$

(c)

(i) $f'(x) = \frac{e^x + e^{-x}}{2} - 1$.

$f''(x) = \frac{e^x - e^{-x}}{2}$.

For all $x > 0, e^x > e^{-x}, \therefore f''(x) > 0$.

(ii) Since $f''(x) > 0$, $f'(x)$ is increasing.

When $x = 0$, $f'(0) = \frac{1+1}{2} - 1 = 0$.

$\therefore f'(x) > 0$ for all $x > 0$.

(iii) Similarly, since $f'(x) > 0$, $f(x)$ is increasing.

$f(0) = \frac{1-1}{2} - 0 = 0$.

$\therefore f(x) > 0$ for all $x > 0$.

$\therefore \frac{e^x - e^{-x}}{2} - x > 0$ for all $x > 0$.

$\therefore \frac{e^x - e^{-x}}{2} > x$ for all $x > 0$.

Q6

(a) The shaded rectangle has sides $2y$ and $(4 - x)$.

$\therefore$ Area $= 2y(4-x)$.

$\therefore \partial V = 2y(4-x)\partial x = 2\sqrt{(4-x)^3}\,\partial x$, since $y = \sqrt{4-x}$.

$$\therefore V = 2\int_0^4 \sqrt{(4-x)^3}\,dx$$

$$= 2\left[\frac{2\sqrt{(4-x)^5}}{-5}\right]_0^4 = \frac{4}{5}\times 32 = \frac{128}{5} \text{ units}^3.$$

(b)

(i) Let the roots be $-1, \alpha$ and β.

$$\prod \alpha = -1\times\alpha\times\beta = -1\left(= -\frac{d}{a}\right).$$

$$\therefore \beta = \frac{1}{\alpha}.$$

(ii) Since all the coefficients are real, $\bar{\alpha}$ is also a root.

$$\therefore \bar{\alpha} = \frac{1}{\alpha}.$$

$\alpha\bar{\alpha} = 1$.

$\therefore |\alpha| = 1$.

(iii) Let $\alpha = a+ib$, where a,b are real.

$\sum\alpha = -1+a+ib+a-ib = 2a-1.$

Since $\sum\alpha = -q$,

$2a-1=-q.$

$a=\frac{1-q}{2}.$

(c)

(i) $PQ^2 = OP^2 - OQ^2 = x^2+y^2-r^2$.

$\therefore PQ = \sqrt{x^2+y^2-r^2}$.

(ii) $PR = |x-c|$

$\therefore \sqrt{x^2+y^2-r^2} = |x-c|.$

$x^2+y^2-r^2 = x^2-2cx+c^2.$

$y^2 = r^2+c^2-2cx.$

(iii) $y^2 = -2c\left(x-\frac{r^2+c^2}{2c}\right).$

$\therefore$ Type $y^2=-4aX$, where $4a=2c$, $\therefore$ focal length $=\frac{c}{2}$.

$\therefore$ Focus $\left(\frac{r^2+c^2}{2c}-\frac{c}{2},0\right)$, which is $\left(\frac{r^2}{2c},0\right)$.

(iv) The directrix has equation $x=\frac{r^2+c^2}{2c}+\frac{c}{2}=\frac{r^2+2c^2}{2c}$.

By definition, $PS=PM$, where M is the foot of P on the directrix.

$\therefore PS = PM = \left|x-\frac{r^2+2c^2}{2c}\right|$

But $PQ=PR=|x-c|.$

$\therefore PS-PQ=\left|x-\frac{r^2+2c^2}{2c}\right|-|x-c|$

$=\frac{r^2+2c^2}{2c}-c=\frac{r^2}{2c}$, which is independent of x.

Q7

(a)

(i) $\ddot{x}=\frac{vdv}{dx}=g-rv$.

$\frac{vdv}{g-rv}=dx.$

$\int\frac{vdv}{g-rv}=\int dx.$

$-\frac{1}{r}\int\frac{g-rv-g}{g-rv}dv=\int dx.$

$-\frac{1}{r}\left(v+\frac{g}{r}\ln(g-rv)\right)=x+C.$

When $x=0, v=0, \therefore C=-\frac{g}{r^2}\ln g.$

$\therefore -\frac{v}{r}-\frac{g}{r^2}\ln(g-rv)+\frac{g}{r^2}\ln g = x.$

$\therefore x=\frac{g}{r^2}\ln\left(\frac{g}{g-rv}\right)-\frac{v}{r}.$

When $x=L$,

$L=\frac{9.8}{0.2^2}\ln\left(\frac{9.8}{9.8-0.2\times 30}\right)-\frac{30}{0.2}=82$ m.

(ii) $v=\frac{dx}{dt}=-\frac{1}{10}e^{-\frac{t}{10}}\left(29\sin t-10\cos t\right)+$

$e^{-\frac{t}{10}}\left(29\cos t+10\sin t\right).$

When $v=0$,

$\frac{29\sin t-10\cos t}{10}=29\cos t+10\sin t.$

$29\sin t-10\cos t=290\cos t+100\sin t.$

$300\cos t=-71\sin t.$

$\tan t=-\frac{300}{71}.$

$t=-1.34$ or $\pi-1.34=1.80.$

When $t=1.80$,

$x=e^{-0.18}\left(29\sin 1.8-10\cos 1.8\right)+92$

$=25.49+92=117.49$ m.

Given that his body length is 2 m, 117.49 + 2 = 119.49 m, $\therefore$ the jumper's head still stays out of the water.

(b)

(i) $z^n=\cos n\theta+i\sin n\theta$.

$z^{-n}=\frac{1}{z^n}=\frac{1}{\cos n\theta+i\sin n\theta}=\frac{\cos n\theta-i\sin n\theta}{\cos^2 n\theta+\sin^2 n\theta}$

$=\cos n\theta-i\sin n\theta.$

Alternatively, $z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$

$=\cos n\theta-i\sin n\theta$, since $\cos(-n\theta)=\cos n\theta$

and $\sin(-n\theta)=-\sin n\theta.$

$\therefore z^n+z^{-n}=2\cos n\theta.$

(ii) $(z+z^{-1})^{2m}=(2\cos\theta)^{2m}.$

$\text{LHS}=z^{2m}+\binom{2m}{1}z^{2m-1}z^{-1}+\binom{2m}{2}z^{2m-2}z^{-2}+$

$+\binom{2m}{m-1}z^{m+1}z^{-m+1}+\binom{2m}{m}z^mz^{-m}+\binom{2m}{m+1}z^{m-1}z^{-m-1}+$

$+\ldots+\binom{2m}{2m-1}z^1z^{-2m+1}+z^{-2m}$

$=\left(z^{2m}+z^{-2m}\right)+\binom{2m}{1}\left(z^{2m-2}+z^{-2m+2}\right)$

$+\binom{2m}{2}\left(z^{2m-4}+z^{-2m+4}\right)+\ldots+\binom{2m}{m-1}\left(z^2+z^{-2}\right)+\binom{2m}{m}$

$=2\cos 2m\theta+\binom{2m}{1}2\cos(2m-2)\theta$

$$+\binom{2m}{2}2\cos(2m-4)\theta+\ldots+\binom{2m}{m-1}2\cos 2\theta+\binom{2m}{m}$$

$$=2\left[\cos 2m\theta+\binom{2m}{1}\cos(2m-2)\theta\right.$$

$$\left.+\binom{2m}{2}\cos(2m-4)\theta+\ldots+\binom{2m}{m-1}\cos 2\theta\right]+\binom{2m}{m}.$$

(iii) $2^{2m}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2m}d\theta=2\left[\frac{\sin 2m\theta}{2m}+\binom{2m}{1}\frac{\sin(2m-2)\theta}{2m-2}\right.$

$$\left.+\binom{2m}{2}\frac{\sin(2m-4)\theta}{2m-4}+\ldots+\binom{2m}{m-1}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$+\left[\binom{2m}{m}\theta\right]_0^{\frac{\pi}{2}}$$

$=\binom{2m}{m}\frac{\pi}{2}$, since sine of a multiple of $\pi=0$.

$$\therefore\int_0^{\frac{\pi}{2}}(\cos\theta)^{2m}d\theta=\frac{1}{2^{2m}}\binom{2m}{m}\frac{\pi}{2}$$

$$=\frac{\pi}{2^{2m+1}}\binom{2m}{m}.$$

Q8

(a)

(i) Let $t=\tan\frac{\theta}{2}$,

$$\cot\theta+\frac{1}{2}\tan\frac{\theta}{2}=\frac{1-t^2}{2t}+\frac{t}{2}=\frac{1-t^2+t^2}{2t}=\frac{1}{2t}$$

$$=\frac{1}{2}\cot\frac{\theta}{2}.$$

(ii) Let $n=1$, LHS $=\tan\frac{x}{2}$,

$$\text{RHS}=\cot\frac{x}{2}-2\cot x=2\cot x+\tan\frac{x}{2}-2\cot x$$

$=\tan\frac{x}{2}$, since from (i), $\frac{1}{2}\cot\frac{x}{2}=\cot x+\frac{1}{2}\tan\frac{x}{2}$,

$$\therefore\cot\frac{x}{2}=2\cot x+\tan\frac{x}{2}.$$

$\therefore$ True for $n=1$.

Assume $\tan\frac{x}{2}+\frac{1}{2}\tan\frac{x}{2^2}+\frac{1}{2^2}\tan\frac{x}{2^3}+\ldots$

$$+\frac{1}{2^{n-1}}\tan\frac{x}{2^n}=\frac{1}{2^{n-1}}\cot\frac{x}{2^n}-2\cot x.$$

RTP: $\tan\frac{x}{2}+\frac{1}{2}\tan\frac{x}{2^2}+\frac{1}{2^2}\tan\frac{x}{2^3}+\ldots$

$$+\frac{1}{2^n}\tan\frac{x}{2^{n+1}}=\frac{1}{2^n}\cot\frac{x}{2^{n+1}}-2\cot x.$$

$$\text{LHS}=\frac{1}{2^{n-1}}\cot\frac{x}{2^n}-2\cot x+\frac{1}{2^n}\tan\frac{x}{2^{n+1}}$$

$$=\frac{1}{2^{n-1}}\left(\cot\frac{x}{2^n}+\frac{1}{2}\tan\frac{x}{2^{n+1}}\right)-2\cot x$$

$$=\frac{1}{2^{n-1}}\times\frac{1}{2}\cot\frac{x}{2^{n+1}}-2\cot x$$

$$=\frac{1}{2^n}\cot\frac{x}{2^{n+1}}-2\cot x=\text{RHS}.$$

$\therefore$ It's true for $n+1$.

Hence, it's true for all $n\geq 1$.

(iii) As $n\to\infty$, $\frac{x}{2^n}\to 0$, $\therefore\frac{\tan\frac{x}{2^n}}{\frac{x}{2^n}}\to 1$.

$$\therefore\frac{1}{2^{n-1}}\cot\frac{x}{2^n}-2\cot x=2\times\frac{\frac{x}{2^n}}{\tan\frac{x}{2^n}}\times\frac{1}{x}-2\cot x$$

$$\to\frac{2}{x}-2\cot x.$$

(iv) $\tan\frac{\pi}{4}+\frac{1}{2}\tan\frac{\pi}{8}+\frac{1}{4}\tan\frac{\pi}{16}+\ldots=\sum_{r=1}^{\infty}\frac{1}{2^{r-1}}\tan\frac{\pi}{2^{r+1}}$

$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{2^{r-1}}\tan\frac{x}{2^r}$, where $x=\frac{\pi}{2}$

$=\lim_{n\to\infty}\frac{1}{2^{n-1}}\cot\frac{\pi}{2^{n+1}}-2\cot\frac{\pi}{2}$, from part (ii)

$=\frac{4}{\pi}-2\cot\frac{\pi}{2}$, from part (iii)

$$=\frac{4}{\pi}.$$

(b) $\frac{1}{n}<\int_{n-1}^{n}\frac{1}{x}dx<\frac{1}{n-1}$.

$$\frac{1}{n}<\left[\ln x\right]_{n-1}^{n}<\frac{1}{n-1}.$$

$$\frac{1}{n}<\ln\frac{n}{n-1}<\frac{1}{n-1}.$$

$$e^{\frac{1}{n}}<\frac{n}{n-1}<e^{\frac{1}{n-1}}.$$

$e^{-\frac{1}{n-1}}<\frac{n-1}{n}<e^{-\frac{1}{n}}$, noting that if $a<b$ then $\frac{1}{b}<\frac{1}{a}$,

$e^{-\frac{n}{n-1}}<\left(\frac{n-1}{n}\right)^n<e^{-1}$, by raising to the power n.

(c)

(i) The probability that a person draws his/her own card is $p=\frac{1}{n}$, $\therefore$ the probability that he/she does not draw his/her own card is $q=1-\frac{1}{n}$.

Pr(A_1 wins in the first draw) $=p$

Pr(A_1 wins in the second draw, i.e. after a round that no one draws own card) $=q^n p$

$\Pr(A_1$ wins in the third draw,i.e. after

two rounds that no one draws own card$) = q^{2n}p$

$\Pr(A_1 \text{ wins}) = p + q^n p + q^{2n} p + ...$

$= \dfrac{p}{1-q^n}$ (using the limiting sum of a GP, since

the ratio $= q^n < 1$)

$\therefore W = \dfrac{p}{1-q^n}$.

$W - q^n W = p$.

$\therefore W = p + q^n W$.

(ii) $W_m = p + q^n p + q^{2n} p + ... + q^{(m-1)n} p$

$= \dfrac{p(1-q^{mn})}{1-q^n}$.

$$\therefore \frac{W_m}{W} = \frac{\dfrac{p(1-q^{mn})}{1-q^n}}{\dfrac{p}{1-q^n}} = 1 - q^{mn}$$

$$= 1 - \left(1 - \frac{1}{n}\right)^{mn}$$

If n is large, $\dfrac{n}{n-1} \approx 1, \therefore$ the result in (b) becomes

$$e^{-\frac{n}{n-1}} \approx e^{-1} < \left(1 - \frac{1}{n}\right)^n < e^{-1}.$$

$$\therefore \left(1 - \frac{1}{n}\right)^n \approx e^{-1}.$$

$$\therefore \left(1 - \frac{1}{n}\right)^{mn} \approx e^{-m}.$$

$$\therefore \frac{W_m}{W} \approx 1 - e^{-m}.$$